Latitude, Longitude, Great Circles

Vocabulary:

geodesic: a curve on a surface giving the shortest distance between two points; different geometries (e.g., Euclidean, spherical, hyperbolic) will have different geodesics.

great circle: the geodesic for a sphere; a circle on the surface of a sphere with maximal circumference.

meridian: on the earth, a north-south great circle (one which connects the poles).

Suppose the earth is a perfectly round ball, which of course it is not, with radius $\rho = 3960$ miles. The angular coordinates θ and ϕ range from 0 to 2π and 0 to π, respectively – except we always work in degrees when thinking about the earth, so really the ranges are $0° \leq \theta \leq 360°$ and $0° \leq \phi \leq 180°$. Those two coordinates give us latitude and longitude, a two-dimensional system for locating points on the surface of the earth.

We think of the equator, $\phi = 90°$, as latitude 0, and have longitude 0 at an arbitrary fixed *prime meridian* (the meridian through Greenwich, England, chosen in 1884 because a majority of ships already used it as the reference meridian on their maps at that time). We talk about latitude and longitude in terms of degrees off of those lines.

For example, Hanover is at latitude 43.42° N and 72.17° W. That is, it is 43.42° north of the equator, so $\phi = 90° - 43.42° = 46.58°$. It is 72.17° west of the prime meridian – but west is clockwise, if you're looking down from the north pole, so its θ is really 72.17° short of a full 360°, or 287.83°.

We can use this information to find the shortest distance between Hanover and other cities. For example, New York City is at latitude 40.75° N and longitude 74° W, giving it spherical coordinates $(3960, 286°, 49.25°)$. To find the distance between Hanover and NYC we think of the earth as centered at the origin. Then vectors of length 3960 in standard position point to spots on the earth. If we convert to Cartesian coordinates, we can use the dot product to find the angle between the vectors (in radians) and multiply by $r = 3960$ to find the arc length of the segment of the great circle connecting the two points.

Hanover's Cartesian coordinates are $(880.70, -2738.14, 2721.87)$, and New York's are $(826.90, -2883.74, 2584.93)$. As we learned long ago, the cosine of the angle between these two points interpreted as vectors is their dot product over the product of their length:

$$\cos\alpha = \frac{\langle 880.70, -2738.14, 2721.87\rangle \cdot \langle 826.90, -2883.74, 2584.93\rangle}{(3960)^2}.$$

Thus $\cos\alpha = 0.9986$, so $\alpha = 0.0529$ radians and the distance is $(0.0529)(3960) = 209.57$ miles.

As a second example, Wellington, New Zealand is at 41.19° S and 174.46° E. South means "past 90°" and east means counterclockwise, so Wellington's spherical coordinates are $(3960, 174.46°, 131.19°)$. Its Cartesian coordinates are $(-2966.10, 287.69, -2607.89)$. The cosine of the angle between the vectors leading to Wellington and Hanover is 0.6695, giving an angle of 2.30 radians and a distance of 9125.15 miles.

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Appendix: Why Are Great Circles Shortest?

It is widely asserted that great circles are geodesics for spheres; that is, that the shortest path along the surface of the sphere joining two points is the shorter piece of the great circle joining those two points. If the points are antipodal, then, there are many shortest paths, but still only one shortest distance. It is less widely *proven* that great circles are geodesics for spheres.

Consider two points P and Q on the sphere. We let ℓ be the arc of a great circle passing through P and Q, and (for contradiction) suppose $\tilde{\ell}$ is another curve on the sphere of strictly shorter length than ℓ. Denote the length of a path by $|\ell|$. We will make an argument which ends by asserting $|\ell| \lneq |\ell|$, and conclude that there can be no such $\tilde{\ell}$.

The first step is to approximate $\tilde{\ell}$ by arcs of great circles. Split $\tilde{\ell}$ into subcurves by choosing points $P_1, P_2, \ldots P_n$ strictly between P and Q which lie on $\tilde{\ell}$, and join adjacent points by arcs of great circles. Name the path you've created ℓ'. If we choose the points close enough together we can get the arc of the great circle between each adjacent pair to be very similar to the portion of $\tilde{\ell}$ joining that pair. In particular, we can get ℓ' close enough to $\tilde{\ell}$ that $|\ell'| \lneq |\ell|$, just as with $|\tilde{\ell}|$.

The next step is to delete the P_i points one by one, starting with P_1. We will let the path ℓ'' be ℓ' from P_2 to Q, but the arc of the great circle joining P and P_2 otherwise. Deleting additional points, we keep replacing the beginning of the path by a great circle arc directly from P to the first remaining point. When all P_i have been removed we are left with ℓ itself, the great circle arc from P to Q. [Or some other great circle arc, if P and Q are antipodal, but the important part is the length, which will be the same.]

The contradiction comes in as follows: we will prove $|\ell''| \leq |\ell'|$ (in particular, still strictly less than $|\ell|$). By extension, then, all the later paths obtained by deleting additional points will be no longer than the path they replace. However, the final path is ℓ itself, and so this will show $|\ell| \lneq |\ell|$, which is a contradiction.

So how do we prove $|\ell''| \leq |\ell'|$? By moving off the sphere and using the fact that great circles lie in planes containing the center of the sphere, which we will call the origin. Since we're talking about ℓ'' in particular, the three great circles we care about are those joining P, P_1, and P_2 in the three possible pairings. The arc lengths will be proportional to the angles at the origin where the planes meet, and the ordinary triangle inequality says the sum of two of those angles must be at least as great as the third one. Therefore ℓ' must be at least as long as ℓ'' from P to P_2, and it is equal otherwise, so $|\ell''| \leq |\ell'|$.